

## Entropy production in the majority-vote model

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We analyzed the entropy production in the majority-vote model by using a mean-field approximation and Monte Carlo simulations. The dynamical rules of the model do not obey detailed balance so that entropy is continuously being produced. This nonequilibrium stochastic model is known to have a critical behavior belonging to the universality class of the equilibrium Ising model. We show that the entropy production exhibits a singularity at the critical point whose exponent is estimated numerically.

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### I. INTRODUCTION

Irreversible systems in the stationary state are in a process of continuous production of entropy. In systems like the one studied here, governed by a master equation, that is, defined by a continuous time Markov process, the irreversibility is characterized by the lack of detailed balance. When a system governed by a Markov process is reversible, that is, when the dynamic rules are such that detailed balance is obeyed, the production of entropy vanishes at the equilibrium state. This is indeed the case of the Glauber model and any other dynamics used to simulate the equilibrium Ising model. The production of entropy is then a signature of irreversibility.

The rate of change of the entropy  $S$  of a system can be properly decomposed into two contributions [1]

$$\frac{dS}{dt} = \Pi - \Phi, \quad (1)$$

where  $\Pi$  is the entropy production due to irreversible processes occurring inside the system and  $\Phi$  is the entropy flux from the system to the environment. The quantity  $\Pi$  is positive definite whereas  $\Phi$  can have either sign. In the stationary state the entropy  $S$  of the system remains constant so that  $\Phi = \Pi$ . Notice that the quantity  $\Phi$  is defined here as the flux from inside to outside the system, so that it will be positive in the stationary state.

In this work we study the steady state production of entropy in a nonequilibrium lattice model, namely, the majority-vote model [2–4]. This is a polling model in which individuals in a community take the opinion of their neighbors with a certain probability  $p$  and the opposite opinion with probability  $q = 1 - p$ . In two or more dimensions, this model displays a continuous phase transition described by the same critical exponents as the equilibrium Ising model [4]. This is in agreement with the conjecture by Grinstein *et al.* [5] according to which nonequilibrium stochastic systems with up-down symmetry fall in the universality class of the equilibrium Ising model.

The flux of entropy is determined by means of an expression which is the average of a function of the rates of transition from one state to another and its reverse [6–9,18]. As the entropy production equals the entropy flux in the stationary state, the former can be determined from the expression for the latter. We use mean-field and Monte Carlo simula-

tions to calculate the entropy flux. In the stationary regime, and at the critical point, the entropy flux, or equivalently, the entropy production, displays a singularity governed by a critical exponent which we estimate numerically.

### II. ENTROPY PRODUCTION

Let us consider a system described by a continuous Markov process with stochastic variables defined over the sites of a regular lattice. A configuration of the system is denoted by  $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_N)$  where  $N$  is the number of sites of the lattice and  $\sigma_i = \pm 1$  is the spin variable associated to site  $i$ . We will be concerned only with one spin flip dynamics, defined by a transition rate  $w_i(\sigma)$  in which the spin variable  $\sigma_i$  changes its sign. The time evolution of the probability  $P(\sigma, t)$  is governed by the master equation,

$$\frac{d}{dt}P(\sigma, t) = \sum_i [w_i(\sigma^i)P(\sigma^i, t) - w_i(\sigma)P(\sigma, t)], \quad (2)$$

where  $\sigma^i = (\sigma_1, \sigma_2, \dots, -\sigma_i, \dots, \sigma_N)$ .

The Gibbs entropy  $S(t)$  of the system at time  $t$  is defined by

$$S(t) = - \sum_{\sigma} P(\sigma, t) \ln P(\sigma, t). \quad (3)$$

Using the master equation (2), its time derivative can be written as

$$\frac{d}{dt}S(t) = \frac{1}{2} \sum_{\sigma} \sum_i \ln \frac{P(\sigma^i, t)}{P(\sigma, t)} [w_i(\sigma^i)P(\sigma^i, t) - w_i(\sigma)P(\sigma, t)]. \quad (4)$$

In agreement with Eq. (1), the right-hand side of this expression should be decomposed into two terms, the entropy production  $\Pi$  and the entropy flux  $\Phi$ . These two quantities have the following expressions [6–9]:

$$\Pi = \frac{1}{2} \sum_{\sigma} \sum_i \ln \frac{w_i(\sigma^i)P(\sigma^i, t)}{w_i(\sigma)P(\sigma, t)} [w_i(\sigma^i)P(\sigma^i, t) - w_i(\sigma)P(\sigma, t)] \quad (5)$$

and

$$\Phi = \sum_{\sigma} \sum_i w_i(\sigma) P(\sigma, t) \ln \frac{w_i(\sigma)}{w_i(\sigma^j)}. \quad (6)$$

The right-hand side of Eq. (5) is always positive as can be easily proved, and the right-hand side of Eq. (6) can be written as the average over the stationary probability distribution, that is,

$$\Phi = \sum_i \left\langle w_i(\sigma) \ln \frac{w_i(\sigma)}{w_i(\sigma^j)} \right\rangle. \quad (7)$$

This is a particularly useful equation because it can be employed to estimate  $\Phi$  from a Monte Carlo simulation. As it is well known only quantities that can be written as averages can be determined numerically in a Monte Carlo simulation. In this sense it is not possible to determine  $S$ , given by Eq. (3), nor  $\Pi$ , given by Eq. (5), but it is actually possible to determine  $\Phi$  from Eq. (6). We remark finally that in the steady state  $\Pi = \Phi$  so that it is possible to determine the entropy production in this regime by a Monte Carlo simulation.

### III. MAJORITY-VOTE MODEL

The majority-vote model is a one-spin-flip stochastic dynamics defined by the following transition rate

$$w_i(\sigma) = \frac{1}{2} \left[ 1 - \gamma \sigma_i \mathcal{F} \left( \sum_{\delta} \sigma_{i+\delta} \right) \right], \quad (8)$$

where  $\mathcal{F}(x)$  is a function that equals  $-1$ ,  $0$ , or  $+1$  according to whether  $x < 0$ ,  $x = 0$ , or  $x > 0$ , and the summation is over the nearest neighbor sites of site  $i$ . Notice that the transition rate  $w_i(\sigma)$  has up-down symmetry, that is, it is invariant under the sign change of all spin variables  $\sigma_i$ . At each time interval, a site  $i$  is chosen at random. If the majority of the neighbors are in state  $+1$  ( $-1$ ) then the site takes the value  $+1$  ( $-1$ ) with probability  $p$  and the opposite sign with probability  $q = 1 - p$  where  $p = (1 + \gamma)/2$ . We will restrict ourselves to the case  $0 \leq q \leq 1/2$  so that  $1 \geq \gamma \geq 0$ . The model can also be interpreted as an Ising system in contact with two heat reservoirs: one, the source of heat, at infinite temperature, and the other, the sink of heat, at zero temperature [4]. Spin systems in contact with two heat baths at different temperatures [10–18] are perhaps the simplest models with nonequilibrium steady states exhibiting dynamic phase transitions.

In the stationary regime, the present model displays a continuous phase transition from an ordered (ferromagnetic) state to a disordered (paramagnetic) state. On a square lattice it is found by numerical simulation that the critical point occurs at  $q_c = 0.075(1)$  [4]. The ordered state occurs when  $0 \leq q < q_c$  and the disordered state when  $q_c < q \leq 1/2$ . For  $0 < q < 1/2$ , this model does not obey detailed balance and we expect a strictly positive entropy production. When  $q = 1/2$  the system is completely disordered and corresponds to a reversible system so that the entropy production vanishes in this case. The critical behavior [4] puts this model in the same universality class as the equilibrium Ising model. This result comes from the conjecture by Grinstein *et al.* [5]

which states that models with stochastic evolution rules with up-down symmetry belong to the Ising universality class.

From the transition rate  $w_i(\sigma)$  given by Eq. (8) it is straightforward to show that

$$B_i(\sigma) = \ln \frac{w_i(\sigma)}{w_i(\sigma^j)} = \left( \ln \frac{q}{p} \right) \sigma_i \mathcal{F} \left( \sum_{\delta} \sigma_{i+\delta} \right). \quad (9)$$

Therefore, the entropy flux per site  $\phi = \Phi/N$  for the majority-vote model can be determined as the average

$$\phi = \langle B_i(\sigma) w_i(\sigma) \rangle. \quad (10)$$

Notice that for a square lattice the function  $\mathcal{F}(x)$  reads

$$\mathcal{F}(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4) = \frac{1}{8} (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4) (3 - \sigma_1 \sigma_2 \sigma_3 \sigma_4). \quad (11)$$

### IV. MEAN-FIELD RESULTS

From the master equation we get the following equations for the time evolution of the magnetization  $\langle \sigma_i \rangle$

$$\frac{d}{dt} \langle \sigma_i \rangle = -2 \langle \sigma_i w_i(\sigma) \rangle. \quad (12)$$

In the first order dynamic mean-field approximation, or simple mean-field approximation, the correlations are neglected and we need only this equation. We apply the approximation to the case of a regular lattice of coordination 4. In the stationary state, the magnetization  $m = \langle \sigma_i \rangle$  is given by the equation

$$m = \frac{\gamma}{2} m (3 - m^2). \quad (13)$$

Using this approximation, we derive the following expression for the entropy flux:

$$\phi = \left( \ln \frac{q}{p} \right) \left( \frac{1}{4} (3m^2 - m^4) - \frac{\gamma}{16} (5 + 6m^2 - 3m^4) \right). \quad (14)$$

The paramagnetic solution,  $m = 0$ , gives the following expression for the entropy flux in the paramagnetic phase:

$$\phi = \frac{5}{16} (1 - 2q) \ln \frac{1 - q}{q}. \quad (15)$$

The ferromagnetic solution is given by the expression

$$m = \sqrt{\frac{1 - 6q}{1 - 2q}}, \quad (16)$$

which is valid for  $q < q_c = 1/6$ . From this result it follows that the entropy flux in the ferromagnetic state is

$$\phi = \frac{q(1 - q)}{1 - 2q} \ln \frac{1 - q}{q}. \quad (17)$$

The stationary entropy flux, or equivalently the entropy production, is a continuous function of the parameter  $q$  as shown

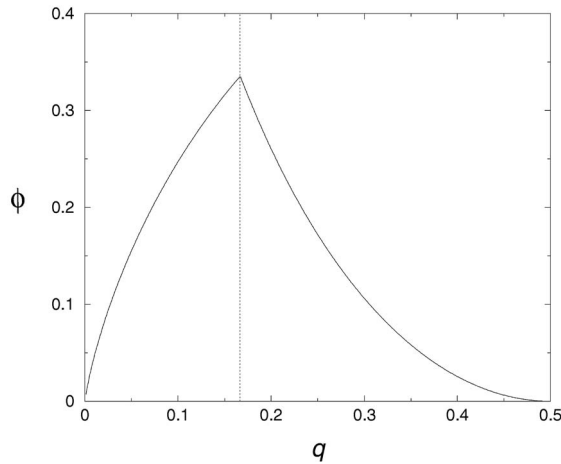


FIG. 1. Entropy flux  $\phi$  for the majority-vote model in the simple mean-field approximation as a function of the external parameter  $q$ . The dotted line indicates the position of the critical point occurring at  $q_c=1/6$ .

in Fig. 1. At the critical point it presents a singularity represented in this mean-field approximation by a discontinuity in the first derivative.

## V. NUMERICAL SIMULATIONS

We have simulated the majority-vote model on a square lattice with periodic boundary conditions for different values of the size  $N=L \times L$  of the system. The simulation was performed as follows. At each time step a site is chosen at random. It takes the value of the majority sign of its neighbors with probability  $p=1-q$  and the opposite sign with probability  $q$  in accordance with the prescription given by Eq. (8). After discarding the first Monte Carlo steps the stationary properties are calculated. We used from  $10^6$  to  $10^7$  Monte Carlo steps to calculate the averages such as the flux  $\phi$  given by Eq. (10). The magnetization and other quantities such as the susceptibility have already been determined by Monte Carlo simulations [4] and will not concern us here. It is found that a continuous phase transition takes place at  $q_c=0.075(1)$  and that the critical exponents are the same as those of the two-dimensional Ising model [4].

In Fig. 2 we show the numerical results for the entropy flux  $\phi$  for several values of the size of the system  $L$ . The entropy flux is finite and continuous. It has a maximum and vanishes when  $q \rightarrow 1/2$  and when  $q \rightarrow 0$  as expected since in these two limits the system reaches an equilibrium stationary state. When  $q \rightarrow 1/2$  we found numerically that the flux vanishes according to

$$\phi = b \left( \frac{1}{2} - q \right)^2, \quad (18)$$

with  $b=0.190(3)$  and when  $q \rightarrow 0$ , it vanishes according to

$$\phi = a q^2 \ln \frac{1-q}{q}, \quad (19)$$

with  $a=1.83(5)$ .

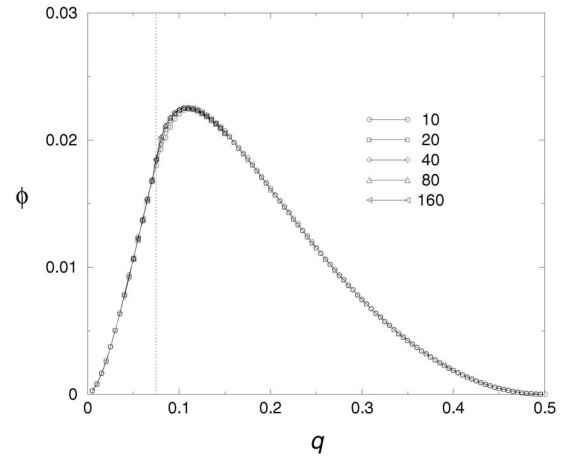


FIG. 2. Stationary entropy flux  $\phi$  for the majority-vote model by Monte Carlo simulation as a function of the external parameter  $q$  for various values of the system size  $L$ . The dotted line indicates the position of the critical point occurring at  $q_c=0.075$ .

We remark that the critical point does not correspond to the maximum of  $\phi$ . Actually, it corresponds to the point of inflexion occurring just before the maximum as can be seen in Fig. 3. At the critical point the entropy flux is finite but has a singularity which we assume to be of the form

$$\phi = \phi_c + A_{\pm} |q - q_c|^{\zeta}, \quad (20)$$

where  $\phi_c$  is the value of the entropy flux at the critical point. The amplitudes  $A_+$  and  $A_-$  correspond to the regimes below ( $q < q_c$ ) and above ( $q > q_c$ ) the critical point. The critical exponent  $\zeta$  should be related to other critical exponents of the majority-vote model. Since the entropy flux is a dynamical quantity,  $\zeta$  may not be related only to the static critical exponents but also to the dynamic critical exponents.

To determine the critical behavior it is convenient to study the derivative of the entropy flux with respect to the external parameter  $q$  which, as follows from Eq. (20), behaves as

$$\frac{d\phi}{dq} \sim |q - q_c|^{\zeta-1}. \quad (21)$$

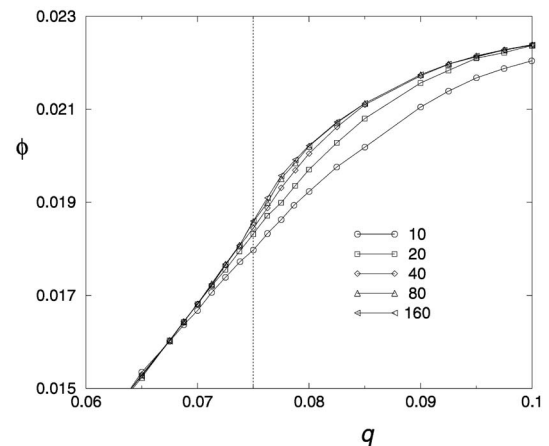


FIG. 3. Enlargement of Fig. 2 around the critical point.

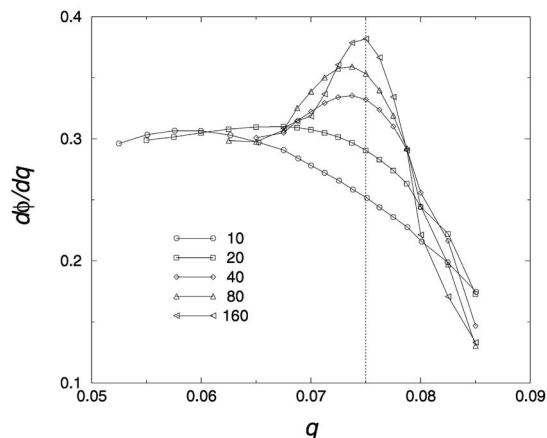


FIG. 4. Derivative  $d\phi/dq$  of the entropy flux  $\phi$  with respect to the external parameter  $q$  for several values of the system size  $L$ . The dotted line indicates the position of the critical point occurring at  $q_c=0.075$ .

We have numerically determined  $d\phi/dq$  for several lattice sizes  $L$  as shown in Fig. 4. Using a finite-size scaling theory [19], then this quantity as a function of  $L$  should diverge at the critical point as

$$\left(\frac{d\phi}{dq}\right)_c \sim L^{(1-\zeta)/\nu} \quad (22)$$

where  $\nu$  is the critical exponent related to the correlation length. In the present case  $\nu=1$  because the majority-vote model is in the universality class of the Ising model. A similar behavior is expected at the maximum. From Fig. 5, we see that the maximum of  $d\phi/dq$  increases approximately as  $\ln L$  leading to an exponent  $\zeta \approx 1$ .

## VI. CONCLUSION

We have determined by mean-field approximation and by Monte Carlo simulations the stationary entropy production of a nonequilibrium model with up-down symmetry, namely,

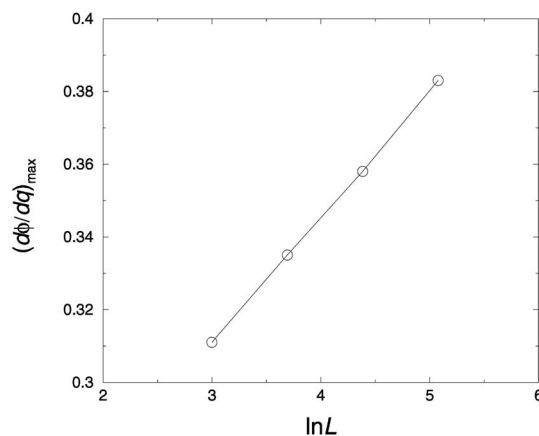


FIG. 5. Maximum of the derivative  $d\phi/dq$  of the entropy flux  $\phi$  with respect to the external parameter  $q$  as a function of the logarithm of the system size  $L$ .

the majority-vote model. The calculation of this quantity by means of Monte Carlo simulations was possible because it equals the entropy flux, in the stationary state, and so this quantity can be written as an average over a stationary probability distribution. The mean-field analysis as well as the Monte Carlo simulations show that the stationary entropy production is positive for nonequilibrium situation, it vanishes when the system attains an equilibrium stationary state, and exhibits a singularity at the critical point. The mean-field results gives a singularity represented by a discontinuous first derivative as usually happens in mean-field calculations of the energy as a function of temperature in equilibrium spin models. The Monte Carlo data, analyzed by a finite-size scaling theory, indicate that the stationary entropy production presents a singularity governed by an exponent  $\zeta \approx 1$ . This exponent may be related to the dynamic exponent. This relationship will be the object of a future investigation.

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